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MATHEMATICS BEYOND THE CALCULUS.

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Students of mathematics generally find that no option as regards the order of subjects is open to them until they have completed an elementary course in calculus. The prescribed road generally is as follows: arithmetic, algebra, geometry, trigonometry, analytic geometry, calculus. After completing an elementary course in calculus the student who expects to become a mathematician frequently finds an entirely different situation since it often becomes necessary for him to decide not only as regards the order of courses but also as to what courses should be selected from the rich offering.

He may perhaps be aided by the fact that there are only three grand divisions of mathematics according to some of the best authorities, even if these divisions do not have any distinct boundaries. They are commonly named as follows: arithmetic and algebra, analysis, geometry. As every mathematician should know something of each one of these great fields, the student will naturally plan to take some comprehensive courses in each of them. To divide the time at his disposal into three approximately equal parts would be easy enough, but to divide it into three parts with due regard to his tastes and ability, and to the men who may be offering the respective courses often becomes a very complex problem.

It is evidently desirable that the student should not decide very early on the particular small field where he hopes to be better informed than any one else. He will probably find many suitable fields without looking for them. In due time he has to make his choice and after it is once made he should bear in mind that serious and honest work is even more important than the choice of the field, for mathematics as a whole will profit by a harmonious development of all the various fields, and those who can appreciate work in but one field are excelled only by the mathematical butterflies in their pernicious influence.

One of the first things that such a student should do is to acquire some fairly intelligent notion as regards the extent of the mathematical literature. The fact that there are about a hundred thousand articles and thirty-five

thousand different books on mathematics, and that such a work as Mueller's *Vokabularium* gives about ten thousand technical mathematical terms may tend to a proper attitude of mind as regards knowing everything and a due appreciation of the need of collaboration. This collaboration naturally calls for leaders of broad sympathies and wide general knowledge, as well as for the more intense workers in limited regions. The rapid expansion of mathematical activity continually demands a greater variety of offorts and hence it offers to the young mathematician greater and greater freedom in selecting his field of usefulness.

Unfortunately it frequently becomes necessary for the student to elect a course about which he knows practically nothing except the name. It is difficult to avoid this. Definitions of a big subject have very little value until after the student has taken a course in the subject, and, strange to say, they frequently become of very little use even when a student knows considerable about the subject. Sometimes teachers, on being examined for public school positions, are asked to define such terms as algebra, and most of them are probably at about the right stage to give definitions. The little they know about the subject can readily be embodied in a definition. On further study, the subject appears to widen so much and to become so interwoven with others that a definition begins to appear hopeless. Hence, if definitions of such mathematical terms are to be given at all they should be given early.

While the student has frequently to begin some courses without much knowledge as regards the nature of the subject which is to be treated, he should aim to get at the sources of information on the subject as early as possible. To do this the collected works of eminent mathematicians are most important. The young mathematician cannot be too familiar with these original sources of information. The bibliographies on the periodical literature are also very helpful. The valuable services which the Royal Society of London Subject Index, the International Catalogue of scientific literature, the Jahrbuch ueber die Fortschritte der Mathematik, and the Revue Semestrielle des publications mathématiques are so well adapted to render should be fully understood, and the student should also acquaint himself, among others, with the following bibliographical aids: The great encyclopedia in German and in French, Hagen's Synopsis der hoeheren Mathematik, Pascal's Repertorium der hocheren Mathematik, Mueller's Fuehrer durch die Mathematische Literatur, Répertoire bibliographique, and Woelffing's Mathematische Buecherschatz. The first and last of these, respectively, cover the periodic and the non-periodic literature of the nineteenth century, the last being, however, very incomplete.

A very practical question is, what courses should be selected as first courses in each of the three great domains of mathematics mentioned above. In view of their extensive applications in other domains the following would be a very suitable set of first courses: theory of numbers, differential equa-

tions, and projective geometry. An almost equally important set of first courses in these three domains would be: theory of discrete groups, functions of a complex variable, and differential geometry. A third important set is higher algebra, functions of a real variable, and algebraic geometry. After completing one or two such general courses in each of these great domains, the student will probably have selected his special subject for thorough study and he may then wisely select more special courses. The number of such special courses that may be open to him will depend more upon the size of the faculty than upon the nature of the subjects. In the great encyclopedia the subject matter of each of the first two grand divisions of mathematics is presented under about thirty general headings, while that of geometry is placed under about forty such headings. As each of these headings is abundantly extensive for a course of lectures, we have here suitable names and material for about one hundred courses, without going outside of the field of pure mathematics and without general considerations of historic and pedagogic subjects.

From this great abundance of material and the tendency to give courses on very special subjects it is evident that the student who would acquire a comprehensive knowledge of mathematics has to do a large amount of reading on subjects which may not be closely related to the courses selected by him. In many of our universities this is especially true as regards the history of mathematics. If mathematicians wish to keep in contact with each other end especially with the larger circle of thinkers who are interested in the general intellectual development, they need a comprehensive view of the history of the various fundamental concepts of mathematics. Such a view is not generally acquired without much effort and the serious student will find that he can only secure a fairly satisfactory knowledge along such general lines by a persistent and wise use ot the moments which are not required for his regular work. Comprehensive historic knowledge will also tend to calmer judgment as regards the ephemeral waves of superficial interest in particular subjects.

We have thus far spoken only of pure mathematics. This is naturally the field of earliest interest as one cannot apply any unknown mathematics, and the applications are frequently more difficult than the considerations as regard ideal conditions of a simple nature. The young mathematicians should however bear in mind that not only ought mathematics be developed harmoniously as a whole, but this development should go hand in hand with the harmonious development of its applications. Sometimes these applications mean only a change of language, and there is no distinct boundary between pure and applied mathematics. Nevertheless, it is of the utmost importance that the mathematician should be able to express his results in the most effective and in the most useful language, and this demands a knowledge of the fundamental laws of the subjects to which mathematical thought may be applied. The wider and more thorough this knowledge the more

probable it will be that the results may be put into the most helpful form. It is in the border land between mathematics and various other sciences where one may reasonably look for the most useful developments, and next to these come the border lands between the various mathematical subjects themselves.

One striking feature of the mathematics beyond the calculus is that, as a rule, it is not arranged in such a carefully graded form as the earlier mathematics. If one considers how few new concepts enter into a course in elementary calculus, or a course in analytic geometry, and how much time and space is used in viewing these concepts from numerous points and how minute most of the thought steps are, one may wonder whether all this was really necessary. As a matter of fact, there is a great waste of time for the brightest student, but so many students of mediocre ability have to get over this ground that these details have been generally adopted.

On the other hand the average ability of the students who go beyond the elementary course in calculus is so very much higher that the thought steps are generally much longer. In fact, they are frequently of such lengths as to discourage the student. This discouragement generally lasts only a short time, and it is frequently replaced by keen enthusiasm, when the student begins to appreciate great thoughts rather than details of calculation, and when he learns to select view-points according to his own taste instead of slavishly following directions. It is true that this requires more time but it also brings better results and prepares the way for independent thought. As all mathematics consists in going from one thought to another near by and repeating this process, it is clear that the power of independent thought is the mathematician's El Dorado.

There is a tendency to speak of the relative importance of work in the different fields of mathematics. In this respect mathematics has much in common with mining. Some of the most enthusiastic reports are based on surface indications or upon developments which are totally inadequate to justify the reports. It is also true that some prospectors seem to lose the ability of passing calm judgments or speaking cautiously. The young mathematician should bear in mind that some of the largest fortunes have been acquired by working low grade ore by improved processes, and that the usual fate of the superficial prospector is not inspiring. In each of the three great grand divisions of mathematics there is in sight an inexhaustible supply of ore of various grades. Some prospecting work in these fields is, however, also highly desirable.

As great theories can be developed only by collaboration it is very important that mathematicians should publish their most useful results from time to time and that they should put them into best possible form. The young mathematician should, however, be very sure of the importance and newness of his results before he offers them for publication. In De Morgan's Budget of Paradoxes, page 4, there occurs the following good advice along this line:

"Most persons must, or at least will, like the lady in Cadogan Place, form and express an immense variety of opinions on an immense variety of subjects; and all persons must be their own guides in many things. So far all is well. But there are many who, in carrying the expression of their own opinions beyond the usual tone of private conversation, whether they go no further than attempts at oral proselytism, or whether they commit themselves to press, do not reflect that they have ceased to stand upon the ground upon which their process is defensible. Aspiring to lead *others*, they have never given themselves the fair chance of being led by *other* others into something better than they can start for themselves; and that they should first do this is what both these classes of *others* have a fair right to expect."

The student should bear in mind that it is necessary for the instructor to express opinions upon an immense variety of subjects and that some instructors seem to think it also necessary to assume the attitude that they know more about every subject under discussion than all the rest of the world put together. Students should not take such attitudes too seriously even if there may be a tendency among their fellow students to accept instructors at the instructors' estimates of their own abilities. Few habits are so harmful to a student as that of a slavish or even a worshipful attitude towards his instructors. On the other hand, he should seek information from many sources outside of the lecture room. If he does this he will become a more independent thinker and lay a much broader foundation for later development.

It should also not be assumed that all the mathematical talent of the world is concentrated in one locality. In mathematics we are still an idolatrous and thoughtless people, and one of the greatest evils we have to contend against is the worship of idols. Let all preach the gospel of only one *mathematics* and that all mathematical worship should be before his throne only. What one individual may do is insignificant when compared with the total development. Much of the time employed in mathematical pilgrimages to Mecca could be more wisely employed. These pilgrimages, however, have spread the contagion of enthusiasm and in this way they have done a great deal of good. The improved facilities as regards communication, especially by means of the mathematical periodicals, make every good library a suitable place for productive mathematical activity; and sound scholarly work should meet, and generally does meet, with a hearty reception on the part of the greatest mathematicians irrespective of the place where the work was accomplished.

In a very general way it may be said that arithmetic and algebra form the basal subjects of mathematics, that analysis is dependent upon arithmetic and algebra, and that geometry depends upon both of the other two grand divisions. The interdependence of these grand divisions is, however, becoming more and more pronounced, and there is a considerable part of geometry which has only a little in common with the other two divisions.

Hence a specialist in geometry may confine himself within just as narrow limits as a specialist in arithmetic and algebra may do. Since analysis is only algebra grown big, it is clear that it is more directly dependent upon the preceding grand division than geometry is.

As mathemathetics largely consists in going from one concept to a related concept, and as concepts which involve the most extensive intimate relations with others are the most helpful, it is clear that it is just as important at times to be able to overlook details as it is at other times to obtain interesting illuminating results from a study of details. For instance, Hamilton, in his lectures on quaternions, makes a distinction between operator and operandus which is unnecessary for applications and is also a hindrance to clear exposition.* One of the great advantages of abstract group theory rests upon overlooking the distinction between operator and operand in many applications. The distinction between a system of conjugate substitutions and a group of permutations as used by Cauchy and Serret is also of doubtful value. The mathematician must be able to leap from mountain top to mountain top as well as to dig out the gold from old river beds concealed by the work of geologic ages.

A CURIOUS MECHANICAL PARADOX.

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Whether a paradox appears merely as a pest depends largely on the point of view. If the paradox lies in some well known and thoroughly accredited discipline such as elementary algebra, geometry, or mechanics, it is certainly a nuisance except in so far as it may be pedagogically instructive for the purpose of reinforcing, by its solution, the very principles which it would upset. To this class belong the demonstrations that $2=4$, usually dependent on a division by 0 or a carelessly placed sign in the extraction of a square root, and the proof that all triangles are isosceles or all angles right angles, dependent on incorrectly drawn figures, and finally the proposition that a rolling wheel must be lighter than one at rest owing to the resultant centrifugal force attributable to the rotation of the wheel about its instantaneous center. Such paradoxes are even a hindrance pedagogically rather than a help unless the fallacy involved can be made much clearer than the fallacious demonstration — a thing almost impossible to accomplish unless the student is so well gounded in the fundamentals of the science that he would not himself fall into like errors. On the other hand, when the paradox arises in a field which is not yet familiar even to specialists in it, a thorough

*E. Study, *Encyklopaedie der Mathematischen Wissenschaften*, vol. I, page 159.